Tentamen Metrische Ruimten
14 april 2009, 09:00 - 12:00 uur, 5111.0022

You can answer the exam in Dutch or English.

1. (a) Consider the topological space H which is the subset $[0,1)$ of $\mathbb{R}$ with the subspace topology. Find the closure of the following sets and whether they are open in H, closed in H, compact, and complete.
   i. $[0,1/2]$,
   ii. $[0,1/2)$,
   iii. $(1/2,1)$,
   iv. $[1/2,1)$.

   (b) Consider the map $p : \mathbb{R} \to [0,1)$ defined by

   $$p(x) = x - [x],$$

   where $[x]$ is the integer part of x, and consider the topological space Q which is the set $[0,1)$ with the topology τ defined by

   $$\tau = \{U \subset [0,1) : p^{-1}(U) \text{ is open in } \mathbb{R}\}.$$

   Show that H in (a) and Q are not topologically equivalent.

2. Consider the function $d : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ defined by

   $$d((x_1,y_1),(x_2,y_2)) = \begin{cases} |y_1 - y_2|, & \text{if } x_1 = x_2, \\ |y_1| + |x_1 - x_2| + |y_2|, & \text{if } x_1 \neq x_2. \end{cases}$$

   Show that d is a metric on $\mathbb{R}^2$. Sketch the open balls $B_1((2,0))$, $B_1((1,2))$, and $B_2((1,1))$.

3. Consider the sequence $f_n : [0,1] \to \mathbb{R}$ of uniformly continuous functions that converges to a function $f : [0,1] \to \mathbb{R}$ with respect to the metric d_∞ given by

   $$d_\infty(f,g) = \sup_{x\in[0,1]} |f(x) - g(x)|.$$

   Show that f is uniformly continuous.

4. Let M be a metric space and H a subset of M. If $f : M \to \mathbb{R}$ and $g : M \to \mathbb{R}$ are continuous functions such that $f(x) = g(x)$ for all $x \in H$, show that $f(x) = g(x)$ for all $x \in Cl(H)$.

Nederlands 2.02.